An efficient polynomial-time approximation scheme for Steiner forest in planar graphs

David Eisenstat[*] Philip Klein[*] Claire Mathieu[*]

Abstract

We give an $O(n \log^3 n)$ approximation scheme for Steiner forest in planar graphs, improving on the previous approximation scheme for this problem, which runs in $O(n^{f(\epsilon)})$ time.

1 Introduction

In the *Steiner forest* problem, we are given an undirected graph G with edge-lengths and a set $\mathcal{D}$ of pairs (s_i, t_i) of vertices. The pairs are called *demands*, and the vertices that appear in demands are called *terminals*. The goal is to find a minimum-length forest F that, for every demand (s_i, t_i), contains a path in F from s_i to t_i. This problem generalizes the Steiner tree problem in networks.

There is a polynomial-time 2-approximation algorithm [1], but the problem doesn't have an approximation scheme unless P=NP [3, 16]. However, for restricted input classes, polynomial-time approximation schemes have been found. For the case where the vertices are the points on the plane and edge-lengths are Euclidean distances, Borradaile, Klein, and Mathieu [5] give an approximation scheme that can be implemented in $O(n \log n)$ time where n is the number of terminals.

For planar graphs, Bateni, Hajiaghayi, and Marx [2] give a polynomial-time approximation scheme. The running time, however, for obtaining a $(1+\epsilon)$-approximate solution has the form $n^{\epsilon^{-c}}$. The degree of the polynomial grows as ϵ gets smaller. An *efficient* polynomial-time approximation scheme is an approximation scheme whose running time has the form $O(f(\epsilon)n^c)$ for some function f and some constant c independent of ϵ. Thus the approximation scheme of Bateni, Hajiaghayi, and Marx is not an efficient PTAS in this sense. Our main result is an efficient PTAS:

Theorem 1.1. *For planar Steiner forest, there is an approximation scheme whose running time is $O(n \log^3 n)$.*

[*]Department of Computer Science, Brown University. Supported in part by NSF grant CCF-0964037.

2 Techniques

2.1 Branchwidth

Tree-decomposition and branch-decomposition are ways to map pieces of the graph to nodes of a tree so that, loosely speaking, pieces have small overlap. We formally define branch-decomposition. The general paradigm is to reduce the problem to graphs of bounded tree- or branch-width.

A *carving* of a ground set is a maximal family $\mathcal{C}$ of mutually noncrossing subsets of the ground set. In this paper, we refer to the sets in $\mathcal{C}$ as *clusters*. The Hasse diagram of the inclusion partial order on the clusters is a tree in which each node has zero or two children.

A *branch-decomposition* of a graph G is a carving $\mathcal{C}$ of the edges of G. The *boundary* ∂C of a cluster C is the set of vertices v such that C contains a proper nonempty subset of the edges incident to v. The *width* of a branch-decomposition $\mathcal{C}$ of G is $\max\{|\partial C| : C \in \mathcal{C}\}$. The *branchwidth* of G is the minimum width over all branch-decompositions of G.

Treewidth (not defined here) is within a constant factor of branchwidth. Graphs of bounded treewidth and branchwidth are tree-like, and many problems can be solved exactly in linear time on such graphs. As we will see, this is *not* the case for Steiner forest.

2.2 Framework

The approximation scheme of Bateni, Hajiaghayi, and Marx fits into the framework of Klein [13], which consists of the following steps:

spanner Find a subgraph G_1 (called the Steiner tree spanner) of the input graph G_0 such that, for constants c and d,

1. $\text{length}(G_1) \leq c\,\text{OPT}(G_0))$, where $\text{OPT}(G_0)$ is the total length of the edges used by an optimum solution, and
2. $\text{OPT}(G_1) \leq (1 + d\epsilon)\,\text{OPT}(G_0)$.

thinning Partition the edges into p subsets such that the contraction of any subset yields a graph of branchwidth $O(p)$. Let G_2 be the graph obtained from G_1 by contracting the subset S having the smallest total length.

dynamic programming Find an (approximately) optimal solution in G_2.

lifting Lift the solution in G_2 to a solution in G_0 by uncontracting edges of S and adding them to the solution as needed.

This presentation of the framework differs from the original in [13] in that, in the original, the dynamic-programming step finds an optimal solution.

Suppose the solution in the dynamic-programming step has length at most $(1 + c'\epsilon)\,\text{OPT}(G_2)$. Since contraction preserves connectivity, $\text{OPT}(G_2) \leq \text{OPT}(G_1)$. By the spanner property, $\text{OPT}(G_1) \leq (1 + d\epsilon)\,\text{OPT}(G_0)$. The addition of some edges from S in the lifting step increases the length by at most $\text{length}(G_1)/p$. We choose $p = c/\epsilon$ so the additional length is at most $\epsilon\,\text{OPT}(G_0)$. Hence the length of the final solution is $((1 + c'\epsilon)(1 + d\epsilon) + \epsilon)\,\text{OPT}(G_0)$.

The dynamic-programming step is straightforward and takes linear time; the construction is given in [13]. (See also [9, 8]. (It has been generalized to bounded-genus graphs [9] and, more recently, to minor-excluded graphs [8].) The lifting step is problem-dependent but straightforward for the problems (such as TSP, Steiner tree, and Steiner forest) to which the framework has been successfully applied. The spanner and dynamic-programming steps are problem-dependent. It is in those steps that Bateni, Hajiaghayi, and Marx [2] introduced new techniques, and it is there that our improvements go.

2.3 Spanner

The spanner step of Bateni et al. built on the same step in the Steiner tree PTAS of Borradaile, Klein, and Mathieu [4, 6]. The proof of the latter can be adapted to show the following. For an instance of Steiner forest consisting of a graph G and a set $\mathcal{D}$ of terminal pairs, let $\text{OPT}(G, \mathcal{D})$ denote the optimum value.

Lemma 2.1 (Borradaile et al., adapted)**.** *For a number $\epsilon > 0$, a planar graph G_0, and a tree T of G_0, there is a subgraph H of length $f(\epsilon)\,\text{length}(T)$ such that, for any set $\mathcal{D}$ of pairs of terminals belonging to T, $\text{OPT}(H, \mathcal{D}) \leq \text{OPT}(G_0, \mathcal{D}) + \epsilon\,\text{length}(T)$, where $f(\cdot)$ is a fixed function. Furthermore, H can be constructed in $O(n \log n)$ time for fixed ϵ.*

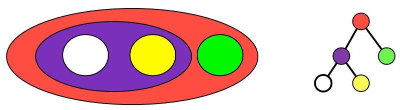
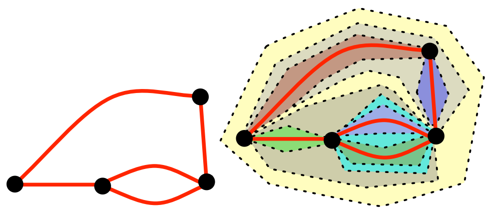

Figure 1: Left: a carving. Right: a branch decomposition.

To use this result, Bateni et al. introduced an algorithm called *prize-collecting (PC) clustering.*

Theorem 2.1 (Bateni et al.). *There is a polynomial-time algorithm that, given a number $\epsilon > 0$ and a (not necessarily planar) Steiner-forest instance $(G, \mathcal{D})$, outputs a partition $\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_\ell$ of $\mathcal{D}$ and corresponding trees $T_1, \ldots, T_\ell$ such that*

1. *the terminals comprising $\mathcal{D}_i$ belong to the tree T_i,*
2. $\sum_{i=1}^{\ell} \text{length}(T_i) \leq (\frac{4}{\epsilon} + 2)\,\text{OPT}(G, \mathcal{D})$, *and*
3. $\sum_{i=1}^{\ell} \text{OPT}(G, \mathcal{D}_i) \leq (1 + \epsilon)\,\text{OPT}(G, \mathcal{D})$.

To obtain a spanner for the Steiner forest instance $(G_{in}, \mathcal{D}_{in})$, therefore, one can perform PC clustering, and then, for each tree T_i, apply Lemma 2.1 to obtain a spanner H_i. The third property of PC clustering implies that the union $\bigcup_{i=1}^{\ell} H_i$ will be a Steiner-forest spanner for the original instance.

Gassner [10] showed that Steiner forest is NP-hard even in graphs of treewidth 3. Bateni et al. addressed this difficulty by giving an (inefficient) PTAS for Steiner forest in bounded-treewidth graphs, one that takes $n^{O(w^2/\epsilon)}$ time (w=width).

2.4 Our improvements to the spanner step

When we try to obtain a quasi-linear approximation scheme, the PC-clustering algorithm of Bateni et al. fails us in two ways.

- The running time is $O(n^2 \log n)$. Indeed, the running time is given in [2] as "polynomial"; beyond that, it does not matter since the overall time for their approximation scheme is $O(n^{f(\epsilon)})$.
- Once the trees $T_1, \ldots, T_k$ are found, a spanner H_i needs to be found for each tree. Finding a spanner, given T_i, takes $O(n \log n)$ time, so the overall time for finding the spanners is $O(kn \log n)$. Since k is $\Omega(n)$ in the worst case, the bound is $O(n^2 \log n)$.

We give a PC-clustering theorem that addresses both issues: the algorithm runs in $O(n \log n)$ for planar graphs (in fact, for any excluded-minor family) and it returns subgraphs $G_1, \ldots, G_\ell$ with small overlap (each edge is in $O(\log n)$ subgraphs) in which the spanners can be found.[1]

Theorem 2.2 (New PC-clustering). *For any $\delta > 0$, there is an algorithm that, given $\epsilon > 0$ and a Steiner forest instance $(G, \mathcal{D})$, outputs a partition $\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_\ell$ of $\mathcal{D}$ and corresponding trees $T_1, \ldots, T_\ell$ and subgraphs $G_1, \ldots, G_\ell$ such that*

1. *the terminals comprising $\mathcal{D}_i$ belong to the tree T_i,*
2. $\sum_{i=1}^{\ell} \text{length}(T_i) \leq (\frac{4+\delta}{\epsilon} + 2)\,\text{OPT}(G, \mathcal{D})$,
3. $\sum_{i=1}^{\ell} \text{OPT}(G_i, \mathcal{D}_i) \leq (1 + \epsilon)\,\text{OPT}(G, \mathcal{D})$,
4. *each edge of G is in $O(\log n)$ of the subgraphs.*

If the input graph G is simple and planar or, more generally, comes from a fixed excluded-minor family, the running time of the algorithm is $O(n \log n)$.

Combining this algorithm with the $O(n \log n)$ construction of Lemma 2.1, we obtain an $O(n \log^2 n)$ algorithm for obtaining a Steiner-forest spanner for simple planar graphs.

PC-clustering is but one example of the use of primal-dual approximation algorithms in approximation schemes for planar, bounded-genus, and minor-excluded graphs. Our technique for speeding up PC-clustering in planar and bounded-genus graphs in such graphs applies to other primal-dual approx-

[1]PC-clustering can be stated in a somewhat more general way and is used in this way in multiterminal cut; our result actually addresses the more general problem.

imation algorithms as well. For example, the technique can be used on the Goemans-Williamson approximation algorithm for prize-collecting TSP and prize-collecting Steiner tree. As a consequence, we obtain $O(n \log n)$ approximation schemes for these problems in planar and bounded-genus graphs. The speed-up in the algorithm comes from use of a dynamic data-structure [7, 14] for maintaining orientations, together with ideas from a data structure [12] for efficient implementation of primal-dual approximation algorithms.

2.5 Our improvements to the dynamic programming step

When we try to obtain an efficient approximation scheme, the dynamic program of Bateni et al. fails us in one way: each tree that crosses the boundary of a cluster is approximately represented by $O(w/\epsilon)$ of its vertices, and there are $n^{O(w/\epsilon)}$ possible such vertex choices.

We take advantage of the spanner property in combination with the bounded branchwidth property. Recall that the graph G_1 has length at most $c\,\mathrm{OPT}(G_0)$, and hence so does the graph G_2 resulting from the thinning step. In Section 4, we prove the following:

Theorem 2.3. *For any constant $\epsilon > 0$, there is an $O(f(w)n \log^2 n)$ algorithm that, for any instance $(G, \mathcal{D})$ of Steiner forest of branchwidth w, finds a solution of length at most $\mathrm{OPT}(G, \mathcal{D}) + \epsilon\,\mathrm{length}(G)$, where $f(\cdot)$ is a fixed function.*[2]

We achieve this using a new graph construction on branch-decompositions. For each cluster, if the sum of lengths of edges near the cluster's boundary is high then the edges are contracted. The result is a graph in which, for each cluster, the sum of lengths of edges near the cluster's boundary is not too big. We can therefore cover the region near the boundary by a constant number of regions of low diameter. This simplifies the dynamic program since it doesn't have to keep track of exactly where the terminals are—just which regions contain them. Since the number of regions is constant, we can get by with fewer configurations.

The situation is a bit more complicated because the dynamic program has to deal with regions at different scales, and has to guess the scales. We show it suffices to guess among a number of scales that is logarithmic in the *height* of the branch-decomposition tree and exhibit a linear-time algorithm that, given an arbitrary branch decomposition of width w, finds a new branch decomposition of width $2w$ and logarithmic height. Another complication is the edges that were contracted in the graph construction. We show that, after uncontracting these edges, the optimal solution can be patched so that its length does not increase much. Consequently, the solution found by the dynamic program has length not much more than optimal.

Combining Theorem 2.2, Theorem 2.3, the $O(n \log n)$ spanner construction of [6] described in Lemma 2.1, and the framework, we obtain Theorem 1.1.

3 Proof of Theorem 2.2

We describe the algorithm that proves Theorem 2.2. It involves just a small change to the PC-clustering algorithm of [2], although our presentation is different.

3.1 Algorithm for basic PC-clustering with graph decomposition

In contrast to [2], we describe PC-clustering using contractions. When an edge uv is contracted, the endpoints are coalesced to form a new vertex. The variable t represents simulated time. Part of the input is an assignment $\phi[\cdot]$ of "energy" to vertices, that, over time,consumed. When two endpoints u, v of an edge are coalesced, the new vertex combines their remaining energy. We say a vertex$v$ is *living* if it has not yet exhausted its energy, i.e. $\phi[v] > 0$ (else *dead*), and $d[v]$ represents the amount of (simulated) time v has lived so far. Our substantive change is to introduce the notion of "zombie" vertices[15], which are vertices that are joined to living vertices not too long (depending on a parameter δ) after they die.

Let G_0 denote the graph G before the contractions of

[2]There is no great significance to our changing from treewidth to branchwidth.

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PC-clustering, Phase 1:
  input: an initial graph $G$ with edge-lengths length$(\cdot)$, and an initial assignment $\phi$ of budgets to vertices
  $t := 0$;   SAVE $:= \emptyset$
  for each vertex $v$, $d[v] := 0$
  while there is a living vertex
    $\Delta_1 := \min\{\phi[v] \ : \ v \in V(G), v \text{ living}\}$
    $\Delta_2 := \min \{\text{length}(uv) \ : \ uv \in E(G), \text{ one of } \{u, v\} \text{ is living}\}$
              $\cup \{\text{length}(uv)/2 \ : \ uv \in E(G), \text{ both } u \text{ and } v \text{ are living}\}$
    $\Delta := \min\{\Delta_1, \Delta_2\}$ # which happens first?
    $t := t + \Delta$ # advance time
    for every living vertex $u$,
      $d[u] := d[u] + \Delta$
      $\phi[u] := \phi[u] - \Delta$
      length$(uv) :=$ length$(uv) - \Delta$ for every incident edge $uv$
    if some edge $uv$ now has zero length,
      contract $uv$, creating new vertex $w$
†     assign $\phi[w] := \phi[u] + \phi[v]$ and $d[w] := \max\{d[u], d[v]\}$
      if some endpoint (say $v$) is not living but $t < (1 + \delta)d[v]$ then add $uv$ to SAVE
  $F_1 :=$ {edges contracted}
PC-clustering, Phase 2:
  initialize $F_2 := F_1$
  while there is an edge $e \in F_2 -$ SAVE that is the only edge incident to a dead vertex $v$
    delete $e$ from $F_2$
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Figure 2: The new PC-clustering algorithm

Phase 1. In the following, unless otherwise stated, the term *vertices* includes the original vertices of G_0 as well as the new ones formed by contraction. For each vertex v, let $\phi_0[v]$ denote the initial value of $\phi[v]$, the value when it is first assigned (whether before Phase 1 commences, in the case of vertices of G_0, or in line † for vertices created by contractions).

The contractions define a binary forest, the *contraction forest*, on the vertices. If an edge uv was contracted and the resulting vertex is w then u and v are the two children of w in the contraction forest.

For each vertex v, let S_v be the set of vertices of G_0 that were coalesced to form v, and let G_v be the subgraph of G_0 induced by S_v.

We say an edge e of G_0 is *incident* to a vertex v if exactly one of the endpoints of e in G_0 belongs to S_v.

A vertex v is *isolated* if, at the end of Phase 2, no edge of F_2 is incident to it. We define the *isolated-dead-vertex forest* $\mathcal{I}$ to be the forest whose nodes are the isolated dead vertices and such that the parent of v is its nearest proper isolated dead ancestor in the contraction forest.

Lemma 3.1. *For each vertex $v \in V(\mathcal{I})$, there is a connected component T_v of F_2 whose vertex set is $S_v - \bigcup\{S_w \ : \ w$ a child of v in $\mathcal{I}\}$.*

Lemma 3.2. *The depth of $\mathcal{I}$ is at most* $1 + \log_{1+\delta} \dfrac{\sum\{\phi_0[u] \ : \ u \in V(G_0)\}}{\min\{\phi_0[v] \ : \ v \in V(G_0), \phi_0[v] > 0\}}$.

Proof. At the end of Phase 1, for each dead vertex v, $d[v]$ is the time when v died. Each root r of $\mathcal{I}$ has $d[r] \leq \sum\{\phi_0[u] \ : \ u \in V(G_0)\}$. Suppose v is a nonroot vertex of $\mathcal{I}$, and let w be the parent of v in $\mathcal{I}$. Let uv be the edge contracted to form w. Since v is isolated, uv does not remain in F_2, so $v \notin$ SAVE. Therefore the time t at which uv is contracted must satisfy $t > (1+\delta)d[v]$. Therefore $d[w] > (1+\delta)d[v]$. This proves the lemma. □

The output of the algorithm is the forest $\mathcal{I}$ and, for each vertex v of $\mathcal{I}$, the subgraph G_v and the connected component T_v of Lemma 3.1. Lemma 3.2 implies that each vertex/edge of G_0 is in a logarithmic number of subgraphs.

3.2 Length of forest returned by basic PC-clustering

Lemma 3.3. *The forest returned has length at most $2(1+\delta)\sum\{\phi_0[v] \ : \ v \in V(G_0)\}$.*

Proof. For each value of t, let Living(t) denote the set of vertices that are living at time t, and let RecentDead(t) denote the set of vertices v such that (time of v's death) $\leq t \leq (1+\delta)$(time of v's death)

For each vertex v, define $\tau[v] = \sum\{\phi_0[u] \ : \ u \in S_v\} - \phi[v]$. Intuitively, $\tau[v]$ is the amount of energy "used up" by v and its descendants in the contraction tree. The algorithm ensures that $\phi[v]$ remains nonnegative, so $\tau[v] \leq \sum\{\phi_0[u] \ : \ u \in S_v\}$. Induction shows $d[v] \leq \tau[v]$.

In Phase 1, when an edge is added to F_1, its its reduced length is zero. The reduction in the length of edge uv can be attributed to the endpoint(s) living.

For time t, let G_t be the graph G at time t, and let H_t be the edge-subgraph of G_t consisting of edges that are in F_2 at the end of Phase 2. The total length of F_2 is at most $\int \sum_{v \in \text{Living}(t)}$(degree of v in H_t) dt. H_t is a forest whose leaves are clusters that are living or recent dead. The degree in H of dead clusters is at least two, so $\sum_{C \in \text{Living}(t)} \deg_H(C) \leq 2(|\text{Living}(t)| + |\text{RecentDead}(t)|)$. Therefore the total length of F_2 is at most $\int 2(|\text{Living}(t)| + |\text{RecentDead}(t)|)\ dt$. While a vertex v is living, $\phi[v]$ is decreasing at unit rate, so $\tau[v]$ is increasing at unit rate. This shows $\int |\text{Living}(t)| dt \leq \sum\{\phi[v] \ : \ v \in V(G_0)\}$. By the definition of RecentDead(t) and $d[v] \leq \tau[v]$ shows $\int |\text{RecentDead}(t)|) dt \leq \delta \sum\{\phi[v] \ : \ v \in V(G_0)\}$. This proves the lemma. □

Remark The only difference between this analysis and that of [2] is the part dealing with RecentDead(t).

Recall that when two vertices coalesce, the resulting vertex gets the remaining energy from its endpoints. Therefore each bit of energy possessed by a new vertex v comes from some original vertex $u \in S_v$. Following [2], we think of the energy originally assigned to u as having the *color* u. If some of v's energy comes from original vertex u, we will say that v has color u. Let E' be a set of edges of G_0. We say a color u is *exhausted* by E' if every vertex v colored by u has an incident edge in E'. These concepts yield:

Lemma 3.4 (Bateni et al.). *Let L be the set of colors exhausted by E'. The length of E' is at least $\sum_{u \in L} \phi_u$.*

Lemma 3.5. *An original vertex u is exhausted by E' if, for some dead vertex v such that $u \in S_v$, E' contains a path between u and some original vertex not in S_v.*

3.3 Using PC-clustering in Steiner forest

Now we prove Theorem 2.2. (The proof of the running time is in Section 3.4.) The input instance of Steiner forest is $(G_{in}, \mathcal{D})$. The algorithm finds a 2-approximation solution F^*, and then obtains the graph G from G_{in} by contracting each connected component K of F^*. Let k be the number of components. Let Y be the set of components of length $< \frac{\epsilon}{2k}\,\text{length}(F^*)$. For each component K not in Y, the algorithm assigns energy to the vertex u of G resulting from contracting K: $\phi[u] := 2\epsilon^{-1}$ length(K). All other vertices of G are assigned zero energy.

The algorithm runs Phase 1 and 2 of Section 3.1 on G and $\phi[\cdot]$, obtaining F_2 and the isolated-dead-vertex forest $\mathcal{I}$. For each $v \in V(\mathcal{I})$, the algorithm obtains a subgraph G_v (see Section 3.1) and (see Lemma 3.1) a connected component T_v of F_2. For each, the algorithm obtains G'_v from G_v and T'_v from T_v by uncontracting the edges of F^*, and defines $\mathcal{D}_v$ to be the set of demands $(s, t) \in \mathcal{D}$ for which $s, t \in V(T'_v)$. We claim that these structures satisfy the conditions in Theorem 2.2. The first condition is satisfied by construction. Lemma 3.3 implies that the second condition is satisfied. Since each vertex of G is initially assigned energy at least $\frac{1}{k}\,\text{length}(F^*)$, Lemma 3.2 implies that $\mathcal{I}$ has depth $\leq 1 + \log_{1+\delta} k$, which implies the fourth condition.

It remains to show the third condition, that the sum of optimum values for the subinstances $\{(G'_v, \mathcal{D}_v) : v \in V(\mathcal{I})\}$ is at most $1 + \epsilon$ times the optimum value for the original instance $(G_{in}, \mathcal{D})$. Since $\mathcal{I}$ is a rooted forest, it induces a partial order on these subinstances.

Let E' be the edge-set of an optimal solution to the original instance. Each connected component C of E' is assigned to the subinstance containing C that is farthest from a root in $\mathcal{I}$, i.e. the instance for which G_v is smallest. Let H_v be the subgraph of G'_v consisting of the components assigned to $(G'_v, \mathcal{D}_v)$.

Since H_v might not constitute a feasible solution for that instance, we might need to augment them. Suppose that there is a demand $(s, t) \in \mathcal{D}_v$ such that s and t are not connected by H_v. The 2-approximate solution F^* contained some connected component K that joined s and t; let u_K be the vertex in G_0 that resulted from contracting that connected component. Since E' is a feasible solution, it too contained a connected component that joined s and t; since that component was not assigned to the subinstance $(G'_v, \mathcal{D}_v)$, it must be that the component is not contained in G'_v, so, by Lemma 3.5, u_K is exhausted by E'. To augment H_v, we add the component K of the 2-approximate solution. In the augmented solution, s and t are joined. Either K belongs to Y or $\phi[u_K] = 2\epsilon^{-1}$ length(K). The sum of the lengths of components in Y is $\leq k \cdot \frac{\epsilon}{2k}\,\text{length}(F^*) \leq \frac{\epsilon}{2}\,\text{length}(E')$, and $\sum\{\phi[u_K] : u_K \text{ is exhausted by } E'\} \leq \text{length}(E')$, so $\sum\{\text{length}(K) : K \notin Y, u_K \text{ is exhausted by } E'\} \leq \epsilon\,\text{length}(E')$. Therefore the sum of lengths of solutions to the subinstances is $\leq (1 + \epsilon)\,\text{OPT}(G_{in}, \mathcal{D})$. This proves the third property of Theorem 2.2.

3.4 Primal-dual on planar and minor-excluded graphs

We show that some primal-dual approximation algorithms, including Goemans and Williamson's approximation algorithm for Steiner forest, and Bateni, Hajiaghayi, and Marx's algorithm for PC clustering (and our modification of this algorithm), can be implemented in $O(n \log n)$ time for planar graphs.

The method is to combine an approach of [12] to implementing primal-dual approximation algorithms with a technique of [7]

3.4.1 Interface to data structure

Klein [12] shows that primal-dual algorithms such as that of [11] can be implemented using a data structure. There are two categories (*active* and *inactive* in the case of primal-dual). An ordered pair (c, c') of categories is called a *bicategory*. Each vertex v is assigned a to category $c(v)$, and thus each edge uv is assigned to a bicategory. The data structure supports the following operations:

- DecreaseCost(b, δ), where b is a bicategory and δ is a real number, decreases by δ the cost of all edges in bicategory b. (2)
- FindMin(b) returns the minimum-cost edge in bicategory b.
- ChangeCategory(v, c) changes the category of v to c (implying changes to the bicategories of edges incident to v).
- ContractEdge(e, c) contracts e and assigns the resulting vertex to category c.

3.4.2 Representation

Now we describe the data structure. We use the ideas of [12] but make some changes to allow the data structure to be made more efficient for graphs from a minor-excluded family.

The data structure maintains the following:

- an orientation of the edges;
- an array $C[\cdot]$, indexed by vertices, such that $C[v]$ is the category of v;
- an array $OUT[\cdot]$, indexed by a vertex v, such that $OUT[v]$ is a linked list of the outgoing edges of v;
- an array $OUT[\cdot,\cdot]$, indexed by a vertex v and a category c, such that $OUT[v, c]$ is a linked list of the outgoing edges of v whose tails are in category c;
- an array $IN[\cdot,\cdot]$, indexed by a vertex v and a category c, such that $IN[v, c]$ is a pointer to a mergeable heap consisting of the incoming edges uv of v for which the tail u has category c;
- an array $B[\cdot]$, indexed by bicategories, such that $B[(c_1, c_2)]$ is a heap consisting of $\{IN(v, c_2) : C[v] = c_1\}$.

Each edge and each heap has a real-number *label*. The data structure maintains the *label invariant*: the cost of an edge is the sum of its label, the label of the heap $IN[v, b]$ that contains it, and the label of the heap $B[b]$ that contains its heap. The key of an edge in the priority queue that contains it is the edge's label. The key of a queue $IN[v, b]$ in the priority queue $B[b]$ that contains it is the label of $IN[v, b]$ plus the minimum key in $IN[v, b]$.

3.4.3 Implementing DecreaseCost and FindMin

DecreaseCost(b, δ) is implemented by decreasing the label of $IN(b)$ by δ. FindMin(b) is implemented by finding the minimum heap in $B(b)$, and returning the minimum edge in that heap.

3.4.4 Implementing ChangeCategory

(v, c) Now we describe how to implement the operation ChangeCategory(v, c). Let $c_0 := C[v]$ (the old category of v). To handle the incoming edges of v, for each category c', the heap $IN[v, c']$ is moved from $B[(c', c_0)]$ to $B[(c', c)]$ (and the label of $IN[v, c']$ is adjusted to preserve the label invariant). To handle the outgoing edges of v, each edge vu in $OUT[v]$ is moved from $IN[u, c_0]$ to $IN[u, c]$ (and the label of vu is adjusted to preserve the label invariant). Time required is $O(|\text{outgoing edges}| \log n)$.

3.4.5 Implementing ContractEdge

To implement ContractEdge(uv, c), first delete the edge uv, and change the categories of u and v to c. Let w denote the vertex to be formed by coalescing u and v. For each category c', merge the heaps $IN[u, c']$ and $IN[v, c']$, and assign the result to $IN[w, c']$; similarly, merge the lists $OUT[u, c']$ and $OUT[v, c']$ and assign the result to $OUT[w, c']$. Remove the edge uv from the heap and list containing it. Update the tables to reflect the fact that u and v no longer exist. The time required is $O((|\{\text{outgoing edges of } u\}| + |\{\text{outgoing edges of } v\}|) \log n)$.

3.4.6 Maintaining bounded outdegree

The time per operation is $O(\log n)$ if the outdegree of each vertex is bounded. Brodal and Fagerberg [7] give a method for dynamically maintaining bounded-outdegree orientations in families of graphs that guarantee the existence of such orientations. Kowalik [14] points out that their method works with contractions. Each update takes amortized $O(\log n)$ time and changes the orientation of $O(\log n)$ edges.

4 Proof of Theorem 2.3

Now we prove Theorem 2.3. We are given an instance $(G_{in}, \mathcal{D}_{in})$ of Steiner forest, and a branch decomposition of G_{in} of width w. For simplicity of presentation, we want to assume that each edge has length 1. To justify this assumption, let $\eta = \epsilon \text{length}(G_{in})/(cm)$, and define a new length assignment $\widehat{\text{length}}(e) := \lfloor \text{length}(e)/\eta \rfloor$. Now all the lengths are integers, and the sum of lengths is at most $c\epsilon^{-1}m$. Replace edge e with $\widehat{\text{length}}(e)$ edgelets (if $\widehat{\text{length}}(e) = 0$ then contract e) to achieve the assumption. Given a solution for the modified instance, the additional length due to rounding is at most ηm, which by definition of η is less than $\epsilon \, \text{OPT}(G_{in}, \mathcal{D}_{in})$.

4.1 Reducing the height of the branch decomposition

Lemma 4.1. *Let $\mathcal{C}$ be a branch decomposition rooted at C of width at most w. The output $\mathcal{C}' =$* Balance($\mathcal{C}$) *is a branch decomposition rooted at C of width at most $2w$, and, for all edges $e \in C$, there exist at most $3\log_2 m + 1$ clusters $D \in \mathcal{C}'$ such that $e \in D$.*

There exist linear-time implementations of Balance *and* Complete.

Proof. For all clusters $D \in \mathcal{C}'$, either $D \in \mathcal{C}$, or there exist clusters $D_1, D_2 \in \mathcal{C}$ with $D_1 \supsetneq D_2$ such that $D = D_1 - D_2$. In the latter case, since $\partial D \subseteq \partial D_1 \cup \partial D_2$ and $|\partial D_1|, |\partial D_2| \le w$, it follows that $|\partial D| \le 2w$.

In order to prove the bound on the number of clusters containing a particular edge, we define a binary function $\delta = \delta(m_1, \ldots, m_k)$. If $m_1 = 1$ and, for all $i \in \{2, \ldots, k\}$, it holds that $m_1 + \cdots + m_{i-1} \ge m_i$, then $\delta = 0$. Otherwise, $\delta = 1$. The use of a heavy path in Balance ensures that every root invocation of Complete has $\delta = 0$.

The bound follows directly from this claim, which we prove by induction on k: for all inputs $\mathcal{C}_1, \ldots, \mathcal{C}_k$ to Complete, for all $i \in \{1, \ldots, k\}$, there exist at most $3(\log_2 m - \log_2 m_i) + 2\delta$ clusters $D \in \mathcal{C}'$ such that $D \supsetneq C_i$. The base case $k = 1$ is trivial, since $\mathcal{C}' = \mathcal{C}_1$. When $k > 1$, let $m_{<j} = m_1 + \cdots + m_{j-1}$ and $m_{>j} = m_{j+1} + \cdots + m_k$ and $\delta_{<j} = \delta(m_1, \ldots, m_{j-1})$. By the choice of j, it holds that $m_{<j} \le m/2$ and $m_{>j} < m/2$. For all $i \in \{j+1, \ldots, k\}$, it follows by the inductive hypothesis that there exist at most $1 + 3(\log_2 m_{>j} - \log_2 m_i) + 2\delta_{>j} < 3(\log_2 m - \log_2 m_i)$ clusters $D \in \mathcal{C}'$ such that $D \supsetneq C_i$. For all $i \in \{1, \ldots, j-1\}$, there exist at most $2 + 3(\log_2 m_{<j} - \log_2 m_i) + 2\delta_{<j} < 3(\log_2 m - \log_2 m_i) + 2\delta$ clusters $D \in \mathcal{C}'$ such that $D \supsetneq C_i$, since $2\delta_{<j} \le 2\delta$. Lastly, there exist at most $2 \le 3(\log_2 m - \log_2 m_j) + 2\delta$ clusters $D \in \mathcal{C}'$ such that $D \supsetneq C_j$, since if $\delta = 0$ then $m_j \le m/2$. $\square$

4.2 A framework for Steiner forest in graphs of bounded branchwidth

In this section, we make explicit the dynamic programming framework of Bateni et al., adapted to branch-decomposition instead of tree-decomposition.

Definition 4.1. *With respect to a cluster C, a vertex $u \in V(C)$ is* active *if it either belongs to ∂C or participates in a demand $\{u, v\} \in \mathcal{D}$ such that $v \notin V(C) - \partial C$ (see Figure 4). We use* active(C) *to denote the active vertices. A demand $\{u, v\} \in \mathcal{D}$ is* active *if either u or v is active.*

Definition 4.2. *Given two partitions P_1 and P_2, let $P_1 \vee P_2$ denote the finest partition coarser than both P_1 and P_2.*

Definition 4.3. *With respect to a cluster C, a* configuration $(\pi^{\text{in}}, \pi^{\text{out}}, \pi^{\text{all}})$ *is a triple consisting of partitions $\pi^{\text{in}}, \pi^{\text{out}}$ of ∂C and a partition π^{all} of* active(C) *such that*

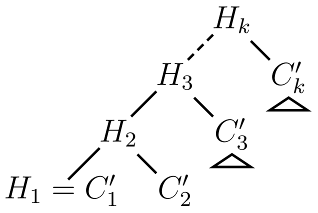
$$\pi^{\text{all}}|_{\partial C} = \pi^{\text{in}} \vee \pi^{\text{out}}.$$

For a subgraph F, in the canonical configuration of *C, π^{in} is the connectivity of ∂C in $F \cap C$, π^{out} is the connectivity of ∂C in $F - C$, and π^{all} is the connectivity of* active(C) *in F (see Figure 5). A subgraph F and a configuration $(\pi^{\text{in}}, \pi^{\text{out}}, \pi^{\text{all}})$ are* compatible *if π^{in} is the connectivity of ∂C in $F \cap C$ and π^{all} is the connectivity of* active(C) *in $(F \cap C) \vee \pi^{\text{out}}$ (see Figure 6). Note that compatibility is determined by the edges in $F \cap C$ only.*

Proposition 4.1. *With respect to each cluster C, a subgraph F is compatible with its canonical configuration.*

Definition 4.4. *Let C_0 be a cluster with child clusters C_1 and C_2. For $i \in \{0, 1, 2\}$, let $(\pi_i^{\text{in}}, \pi_i^{\text{out}}, \pi_i^{\text{all}})$ be a configuration with respect to C_i.*

```
BALANCE($\mathcal{C}$):
Input: a branch decomposition $\mathcal{C}$ with root cluster $C$.
Output: a balanced branch decomposition $\mathcal{C}'$ with root cluster $C$.
Let $H_1 \subsetneq H_2 \subsetneq \cdots \subsetneq H_k = C$ be a heavy path, that is,
    a maximal ascending chain such that, for all $i \in \{1, \ldots, k-1\}$,
    the sibling $H_{i+1} - H_i$ of $H_i$ satisfies $|H_{i+1} - H_i| \leq |H_i|$.
Let $C'_1 = H_1$. For $i \in \{2, \ldots, k\}$, let $C'_i = H_i - H_{i-1}$.
For $i \in \{1, \ldots, k\}$, let $\mathcal{C}'_i = $ BALANCE($\{D : D \in \mathcal{C},\ D \subseteq C_i\}$).
Let $\mathcal{C}' = $ COMPLETE($\mathcal{C}'_1, \ldots, \mathcal{C}'_k$).
```



```
COMPLETE($\mathcal{C}_1, \ldots, \mathcal{C}_k$):
Input: branch decompositions $\mathcal{C}'_i$ with root clusters $C'_i$.
    Clusters $C'_1, \ldots, C'_k$ are pairwise disjoint.
Output: a branch decomposition $\mathcal{C}' \supseteq \mathcal{C}'_1 \cup \cdots \cup \mathcal{C}'_k$
    with root cluster $C'_1 \cup \cdots \cup C'_k$.
For $i \in \{1, \ldots, k\}$, let $m_i = |C'_i|$.
Let $m = m_1 + \cdots + m_k$.
Find (via binary search) the least index $j \in \{1, \ldots, k\}$
    such that $m_1 + \cdots + m_j > m/2$.
If $j > 1$:
    Let $\mathcal{A} = $ COMPLETE($\mathcal{C}'_1, \ldots, \mathcal{C}'_{j-1}$).
    Let $\mathcal{B} = \mathcal{A} \cup \{C'_1 \cup \cdots \cup C'_j\} \cup \mathcal{C}'_j$.
Else:
    Let $\mathcal{B} = \mathcal{C}'_j$
If $j < k$:
    Let $\mathcal{D} = $ COMPLETE($\mathcal{C}'_{j+1}, \ldots, \mathcal{C}'_k$).
    Let $\mathcal{C}' = \mathcal{B} \cup \{C\} \cup \mathcal{D}$, where $C = C'_1 \cup \cdots \cup C'_k$.
Else:
    Let $\mathcal{C}' = \mathcal{B}$
```

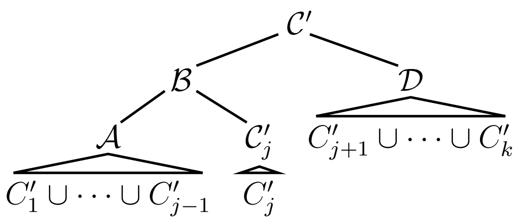


Figure 3: Algorithm for balancing a branch decomposition.

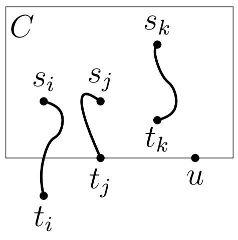


Figure 4: Active vertices. Consider cluster C and demand pairs $\{s_i, t_i\}$, $\{s_j, t_j\}$ and $\{s_k, t_k\}$. Here, s_i, s_j, t_j and u are active vertices of C but t_i, s_k, t_k are not.

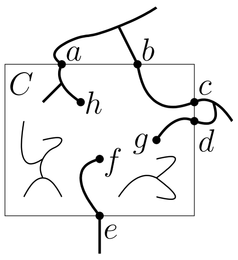


Figure 5: Canonical configuration. Consider the subgraph F depicted above, and assume that a, b, c, d, e, f, g are the active vertices of cluster C. Then the canonical configuration of C for F is: $\pi^{\text{in}} = \{a\}, \{b, c\}, \{d\}, \{e\}$, $\pi^{\text{out}} = \{a, b\}, \{c, d\}, \{e\}$, and $\pi^{\text{all}} = \{a, b, c, d, g, h\}, \{e, f\}$.

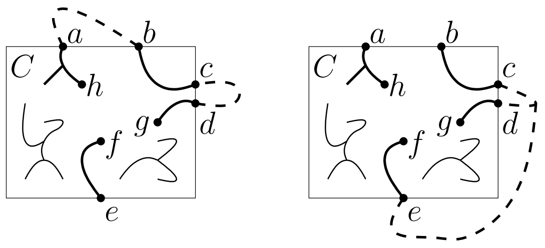


Figure 6: Compatibility of a subgraph with a configuration. If the set of active vertices is $\{a, b, c, d, e, f, g, h\}$, then the subgraph F depicted above is compatible with the configuration $\pi^{\text{in}} = \{a\}, \{b, c\}, \{d\}, \{e\}$, $\pi^{\text{out}} = \{a, b\}, \{c, d\}, \{e\}$, and $\pi^{\text{all}} = \{a, b, c, d, g, h\}, \{e, f\}$ (left side). It is also compatible with the configuration $\sigma^{in} = \pi^{\text{in}}$, $\sigma^{out} = \{a\}, \{b\}, \{c, d, e\}$, and $\sigma^{all} = \{a, h\}, \{b, c, d, e, f, g\}$ (right side).

The configurations $(\pi_0^{\text{in}}, \pi_0^{\text{out}}, \pi_0^{\text{all}})$, $(\pi_1^{\text{in}}, \pi_1^{\text{out}}, \pi_1^{\text{all}})$ *and* $(\pi_2^{\text{in}}, \pi_2^{\text{out}}, \pi_2^{\text{all}})$ *are* compatible *if all of the following conditions hold.*

- $\pi_0^{\text{in}} = (\pi_1^{\text{in}} \vee \pi_2^{\text{in}})|_{\partial C_0}$*: the internal connectivity of the parent is the join of the internal connectivity of the children.*
- *For* $i \in \{1, 2\}$*, it holds that* $\pi_i^{\text{out}} = (\pi_0^{\text{out}} \vee \pi_{3-i}^{\text{in}})|_{\partial C_i}$*: the external connectivity of a child is the join of the external connectivity of the parent and the internal connectivity of the other child.*
- *For* $i \in \{0, 1, 2\}$*, it holds that* $\pi_i^{\text{all}} = (\pi_0^{\text{all}} \vee \pi_1^{\text{all}} \vee \pi_2^{\text{all}})|_{\text{active}(C_i)}$.

See Figure 7 for an example.

Lemma 4.2. *Let F be a subgraph. Then, for every cluster C_0 with child clusters C_1 and C_2, the canonical configurations of F with respect to C_0, to C_1 and to C_2 are compatible.*

Conversely, suppose that we have a configuration for each cluster, such that for every cluster C_0 with child clusters C_1 and C_2, the configurations for C_0, for C_1 and for C_2 are compatible. Then there exists a subgraph F such that the configurations are the canonical configurations of F with respect to the clusters.

The configurations essentially give a local representation of F.

We now show how to use the representation of F with configurations to determine, with local conditions, whether F is a feasible Steiner forest solution.

Definition 4.5. *With respect to a cluster C, a connected component is* outgoing *if it intersects ∂C. A configuration is* outgoing *if every part of π^{all} intersects ∂C.*

Proposition 4.2. *If F is a Steiner forest solution, then, with respect to a cluster C, for each active vertex u, the tree of F containing u is outgoing.*

Definition 4.6. *Let C_0 be a cluster with child clusters C_1 and C_2. Three configurations for C_0, C_1, C_2 are* demand-consistent *if they are compatible, outgoing, and if the following condition holds in addition: For all demands $\{s, t\}$ active for C_1 and C_2 but not C_0, terminals s and t are related by $\pi_0^{\text{all}} \vee \pi_1^{\text{all}} \vee \pi_2^{\text{all}}$.*

For example, in the example of Figure 7, the configurations are demand-consistent because terminal g is related to terminal g': in π_1^{all}, g is connected to d, and in π_2^{all}, d is connected to g'. For an example where π_0^{all} comes into play, see Figure 8.

The following lemma establishes that a subgraph represented by its canonical configurations is a feasible solution if and only if the configurations are consistent with one another.

Lemma 4.3. *A subgraph F is a Steiner tree solution if and only if, for every cluster C_0 with children C_1, C_2, the canonical configurations of F with respect to C_0, C_1 and C_2 are demand-consistent.*

Now, suppose that for each cluster $\mathcal{C}$, we restrict attention to a subset $\Pi_{\mathcal{C}}$ of the configurations of $\mathcal{C}$. We call those configurations *simple*. Then the above setup leads to a dynamic program to find the shortest Steiner forest F such that for every cluster $\mathcal{C}$, the canonical configuration of $\mathcal{C}$ for F belongs to $\Pi_{\mathcal{C}}$. The dynamic program works as follows.

```
For each cluster $C_0$ in bottom-up order,
  if $C_0$ is a single edge $e$,
    then the cost of a configuration is either 1 or 0
      depending on whether $e$ needs to be in.
  else, let $C_1$ and $C_2$ denote the two children of $C_0$;
    for each configuration $\pi_0 \in \Pi_{C_0}$,
      $\text{cost}(\pi_0) := \min(\text{cost}(\pi_1) + \text{cost}(\pi_2))$,
        where the min is over $\pi_1 \in \Pi_{C_1}$ and $\pi_2 \in \Pi_{C_2}$
        such that $\pi_0, \pi_1, \pi_2$ are demand-consistent.
```

The runtime of that dynamic program is $O(n(\max_{\mathcal{C}} |\Pi_{\mathcal{C}}|)^3)$, times the cost of checking that three configurations are demand-consistent.

Bateni et al. proposed a definition of $\Pi_{\mathcal{C}}$ such that $\max_{\mathcal{C}} |\Pi_{\mathcal{C}}| = n^{\text{poly}(1/\epsilon)}$, and proved that, under their definition, there exists a near-optimal forest such that for every $\mathcal{C}$, the canonical configuration of $\mathcal{C}$ belongs to $\Pi_{\mathcal{C}}$. Here, building on the spanner property, we propose a different definition of $\Pi_{\mathcal{C}}$, such that $\max_{\mathcal{C}} |\Pi_{\mathcal{C}}| = O((\log \log n)^{f(\epsilon)})$. This gives us the improvement from an inefficient to an efficient approximation scheme.

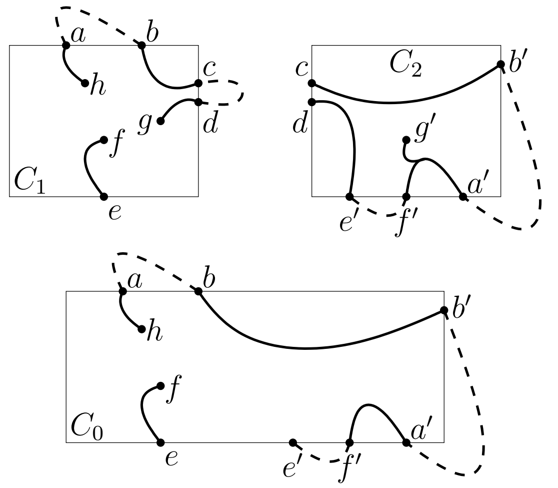


Figure 7: Compatible configurations. In the above example, assume that g is active for C_1 and g' is active for C_2, but neither g nor g' are active for C_0. The dashed connections represent π_i^{out}, the solid connections determine π_i^{in}, and the union of solid and dashed edges determine π_i^{all}.

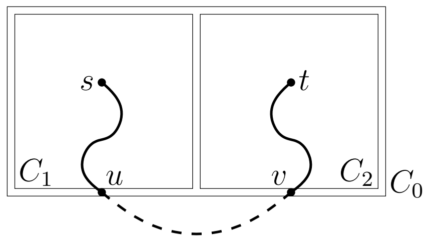


Figure 8: s and t are related by $\pi_0^{\text{all}} \vee \pi_1^{\text{all}} \vee \pi_2^{\text{all}}$: s is connected to u via π_1^{all}, u is connected to v via π_0^{all}, and v is connected to t via π_2^{all}.

In a nutshell, here is the idea: if the optimal forest has several trees that come close to the same vertex v of ∂C, then we connect them with paths to transform them into a single tree; that simplifies the configuration. We charge the length of that path to the total length of edges in the neighborhood of v in ∂C. To make sure that we do not charge the same edges several times over, we contract edges that get charged.

4.3 Contractions in a branch decomposition

For a graph G and a set S of vertices, $SP(G, S)$ denotes the[3] shortest-path forest rooted at S. For a number k, define $SP(G, S, k) = \{e \in SP(G, S) :$ e is in a path of length at most k starting at $S\}$. That is, $SP(G, S, \eta)$ is the shortest-path forest for vertices of G whose distance from ∂G is at most η.

Let G be a graph with branch decomposition $\mathcal{C}$. Fix a parameter α whose value, a function of ϵ and of w, will be set later. We define a recursive algorithm that operates on the clusters of $\mathcal{C}$. For each cluster C in bottom-up order, it computes a radius ρ^C, such that paths of length ρ^C are much shorter (by a factor of α) than the number of edges within distance ρ^C of ∂C. It then contracts all edges within distance ρ^C of ∂C.

```
def CONTRACT$_\alpha(C)$:
    $A^C = \bigcup_{C_i \text{ child of } C} \text{CONTRACT}_\alpha(C_i)$
    $\rho^C := \max\{\rho \ : \ \text{length}(SP(C/A^C, \partial C, \rho)) \geq \alpha\rho\}$
    $B^C = SP(C/A^C, \partial C, \rho^C) \cup A^C$
    return $B^C$
```

What are the structural properties achieved by the contraction algorithm? First, the contracted graph C/B^C has linear growth rate:

Lemma 4.4. *For every $\rho \geq 0$, $\text{length}(SP(C/B^C, \rho)) \leq \alpha\rho$.*

Proof. Observe the following simple property of contractions: $SP(G, S, \eta_1 + \eta_2)$ is the disjoint union of $SP(G, S, \eta_1)$ and $SP(G/SP(G, S, \eta_1), S, \eta_2)$. Thus, $\text{length}(SP(C/B^C, \partial C, \rho)) = \text{length}(SP(C/A^C, \partial C, \rho^C + \rho)) - \text{length}(SP(C/A^C, \partial C, \rho^C))$. By definition of ρ^C, $SP(C/A^C, \partial C, \rho^C)$ has total length at least $\alpha\rho^C$. By maximality of ρ^C, the length of $SP(C/A^C, \partial C, \eta^C + \rho)$ is less than $\alpha(\eta^C + \rho)$. The lemma follows. □

Second, the sum of all radii of contracted areas is small compared to the total length of G:

Lemma 4.5. $\sum_{C \in \mathcal{C}} \rho^C \leq \text{length}(G)/\alpha$.

Proof. By definition, $\rho^C \leq \text{length}(SP(C/A^C, \partial C, \rho^C))/\alpha$. By definition of B^C, $\text{length}(SP(C/A^C, \partial C, \rho^C)) = \text{length}(B^C) - \sum_{C_i \text{ child of } C} \text{length}(B^{C_i})$. Summing over clusters $C \in \mathcal{C}$ gives the lemma. □

What is the running time of the contraction algorithm? Let us explain in more detail how to compute ρ^C efficiently. Here, for each vertex, $d[u]$ denotes the distance from ∂C to u:

```
for $i := 1, 2, 3, \ldots,$
    $s_i := \text{length}(\{uv \in SP(C/A^C, \partial C) \ : \ d[u] = i - 1,$
            $d[v] = i\})$
$\rho^C := \max\{i \ : \ s_1 + \cdots + s_i \geq \alpha\, i\}$
```

The total time for each invocation $\text{CONTRACT}_\alpha(C)$ is linear in the number of edges of C. By Lemma 4.1, $\mathcal{C}$ is a log-height branch-decomposition, so the total time for calling CONTRACT_α on the root cluster of a graph G of $O(n)$ edges is $O(n \log n)$.

4.4 Regions covering partially contracted clusters

Here is a high-level description of our method for finding regions for a cluster C. Fix a parameter β whose value, a function of ϵ and of w, will be set later.

[3]For uniqueness, assume the edges of G are assigned distinct ID numbers, and define $SP(G, S)$ to be the shortest-path forest in which ties are broken by ID number.

```
Denote by $2^{\mu^C}$ the minimum power of two
  greater than $\max_{u\in \text{active}(C)} \text{dist}(u, \partial C)$.
For each $i = 1, 2, \cdots, \mu^C$,
  define a set $\mathcal{L}_i$ of regions of $C/B^C$ such that
    each region has diameter at most $\beta 2^i$, and
    together, the regions of $\mathcal{L}_i$ cover $SP(C/B^C, 2^i)$.
```

It follows from Lemma 4.4 that a greedy algorithm produces such a covering, of size $|\mathcal{L}_i| = O(\alpha/\beta + |\partial C|)$. However, in order to get near-linear running time, we need an algorithm that is slightly more sophisticated than greedy. In the rest of this subsection, we present the details of the algorithm.

For each tree of $SP(C/B^C, \mu^C)$, let v be a tree vertex that is on the boundary ∂C and let $\mathcal{T}_v$ denote the sequence of vertices encountered on an Euler tour of the tree. An *i-region* is a subpath of $\mathcal{T}_v^C$ of length at most $\beta 2^i$ whose first vertex is at a distance at most $2^i + \beta 2^i$ from ∂C in C/B^C. In our algorithm, $\mathcal{L}_i$ is a set of i-regions. For a vertex u of C, we use $d[u]$ to denote the ∂C-to-u distance in C/B^C.

```
for $i := 1, 2, \ldots, \mu^C$,
  $\mathcal{L}_i := \emptyset$
for each tree of $SP(C/B^C, \mu^C)$,
  root the tree at some vertex $v \in \partial C$
  construct an Euler tour $T_v$ of the tree
  for $i := \mu^C, \mu^C - 1, \ldots, 1$,
    $S_{v,i} := \{j\beta 2^i \ : \ j \in \{0, 1, 2, 3, \ldots\}$
                 and $d[T_v[j\beta 2^i]] \le (1+\beta)2^i\}$
    for each $x \in S_{v,i}$,
      $\mathcal{L}_i := \mathcal{L}_i \cup$
        {subpath of $T_v$ of length $\beta 2^i$
        truncated at $|T_v|$) starting at $T_v[x]$}
```

Lemma 4.6. *For each $i = 1, 2, \cdots, \mu^C$, the regions of $\mathcal{L}_i$ cover $SP(C/B^C, 2^i)$.*

Lemma 4.7. *For each $i = 1, 2, \cdots, \mu^C$, $|\mathcal{L}_i| \le 2\alpha(1+2\beta)\beta^{-1} + |\partial C|$.*

Proof. If a vertex u is in a subpath of $\mathcal{L}_i$, then it is at distance at most $\beta 2^i$ from the starting point of the subpath, and so $u \in SP(C/B^C, (1+2\beta)2^i)$. So, the sum of lengths of the i-regions is at most the length of the Euler tours of $SP(C/B^C, (1+2\beta)2^i)$, which is at most twice the number of edges in $SP(C/B^C, (1+2\beta)2^i)$. By Lemma 4.4, $|SP(C/B^C, (1+2\beta)2^i)|$ is at most $\alpha(1+2\beta)2^i$. Each i-region constructed from T_v has length exactly $\beta 2^i$, except possibly the last one. The lemma follows. $\square$

The time for finding the covers is $O(n \log n)$.

4.5 Simple configurations

Definition 4.7. *Fix a cluster C with boundary vertices ∂C and, for each $i = 1, 2, \cdots, \mu^C$, a covering $\mathcal{L}_i$ of C/B^C by i-regions. Given*

- *an integer $d \le |\partial C|$;*
- *d powers of 2 in the range $[2^{\mu^C}/(2\gamma), 2^{\mu^C}]$, where γ is a parameter to be determined later;*
- *a priority ordering over those d numbers, labelled $(2^{i_1}, 2^{i_2}, \ldots, 2^{i_d})$ by order of priority; and*
- *for each i_j, a set of i_j-regions $\mathcal{Q}_j \subseteq \mathcal{L}_{i_j}$;*

consider the subpartition of active(C), *denoted $P(i_1, \ldots, i_d, \mathcal{Q}_1, \ldots, \mathcal{Q}_d)$, and defined by greedily setting the jth part to be*

$$P_j = \big(\text{active}(C) \cap \textit{Uncontract}(\bigcup_{Q \in \mathcal{Q}_j} Q)\big) - \bigcup_{\ell=1}^{j-1} P_\ell,$$

where Uncontract takes as input vertices of C/B^C and outputs the corresponding vertices of C. A configuration $(\pi^{\text{in}}, \pi^{\text{out}}, \pi^{\text{all}})$ is simple *iff $\pi^{\text{all}} = \pi^{\text{in}} \vee \pi^{\text{out}} \vee P(i_1, \ldots, i_d, \mathcal{Q}_1, \ldots, \mathcal{Q}_d)$ for some $(i_1, \ldots, i_d, \mathcal{Q}_1, \ldots, \mathcal{Q}_d)$.*

To understand this definition intuitively, d should be interpreted as the number of outgoing trees. The d powers of 2 should be interpreted as the approximate "radii" of those trees – maximum distance from ∂C to an active tree vertex. As in the algorithm of Bateni et al., the ordering should be interpreted as giving priority to trees whose minimal enclosing cluster is smaller. As in the algorithm of Bateni et al., the (uncontracted) i_j-regions should be interpreted as a covering of the active vertices of the jth tree.

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For each cluster $C$
  define a table indexed by $(i_1, \ldots, i_d, \mathcal{Q}_1, \ldots, \mathcal{Q}_d)$
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Lemma 4.8. *Taking α and β to be constant, the number of simple configurations is $(\log_2 \gamma)^{f(\epsilon)}$ for some function f of ϵ. The time to check demand-consistency is $O(\log n)$.*

We can finally state the main structural Theorem that is at the core of Theorem 2.3.

Theorem 4.1. *For any solution F, there exists a solution $F' \supseteq F$ such that for every cluster C, the canonical configuration of F' with respect to C is simple, and whose length satisfies:* $\text{length}(F') \leq \text{length}(F) + 4\beta(2w-1)\big(1 + (3\log_2 m + 1)/\gamma\big)\,\text{length}(F) + 2\alpha^{-1}(2w-1)\,\text{length}(G)$.

4.6 Proof of Theorem 4.1

4.6.1 Defining F'

F' is simply an extension obtained from F by adding some edges. First, given F and a cluster C, we define a partition of the form $P(i_1, \ldots, i_d, \mathcal{Q}_1, \ldots, \mathcal{Q}_d)$.

Let d be the number of outgoing trees of F. Label these trees $T_1, \ldots, T_d$ in a particular order, such that the following property holds: if the minimal cluster including $E(T_j)$ is a proper descendant of the minimal cluster including $E(T_\ell)$, then $j < \ell$. We now define $i'_j = \lceil \log_2 \max_{u \in V(T_j) \cap \text{active}(C)} \text{dist}(u, \partial C) \rceil$ and $i_j = \max\{i'_j,\ \mu - \lceil \log_2 \gamma \rceil - 1\}$ and let $\mathcal{Q}_j \subseteq \mathcal{L}_j$ be a minimal set of regions such that $\bigcup_{Q \in \mathcal{Q}_j} Q$ covers the vertices of $(T_j \cap C)/B^C$. This defines the radii and sets of regions, hence also specifies the associated partition $(P_1, \ldots, P_d)$ of active(C).

The construction of F' is in two steps. First we go from F to F_1 by adding some edges, then we go from F_1 to F' by adding more edges.

First step: starting from $F_1 := F$, modify F_1 by processing clusters $C \in \mathcal{C}$ in top-down order. Consider a cluster C. While there exists a j and an active vertex u that is in P_j but is not connected to T_j/B^C in F_1/B^C, add to F_1 a path of C/B^C connecting u to T_j/B^C. (This first step is similar to the construction in Bateni el al.)

Second step: starting from $F' := F_1$, modify F' by processing clusters $C \in \mathcal{C}$ in top-down order. Consider a cluster C. While there exists a pair of boundary vertices $u, v \in \partial C$ such that u and v are not connected in $(F' \cap C)/A^C$ but can be connected by adding at most $2\rho^C$ edges of C/A^C to F', add those edges to F'.

The result of this processing defines F'.

4.6.2 The canonical configuration of F' is simple

Lemma 4.9. *After the first step of the construction, for every cluster C we have: all active vertices of P_j are connected to T_j in $(F_1 \cap C)/B^C$.*

Lemma 4.10. *After the second step of the construction, for every cluster C we have: if two vertices of ∂C are connected in $(F' \cap C)/B^C$ then they are connected in $F' \cap C$.*

Proof. It suffices to show that, after those paths are added, if two vertices of ∂C are connected in $(F' \cap C)/B^C$ then they are connected in $(F' \cap C)/A^C$. Suppose not, and let p be a ∂C-to-∂C path in $(F' \cap C)/B^C$ that is not a path in $(F' \cap C)/A^C$, chosen so as to have a minimal number of edges. Then, in C/A^C, p starts at some vertex p_{start} of $SP(C/A^C, u, \rho^C)$ and ends at some vertex p_{end} of $SP(C/A^C, v, \rho^C)$. Concatenating p with a path from u to p_{start} at one end, and with a path from p_{end} to v at the other end, gives a path from u to v in C/A^C. The total number of edges thus added is at most $2\rho^C$, so it would have been added during the processing. □

Lemma 4.11. *For every cluster C, the canonical configuration of F' is simple.*

Proof. It suffices to show for all j that all active vertices in P_j are connected by F' to T_j. Let u' be an active vertex in P_j. Either in F u' belongs to T_j, in which case there is nothing to show, or u' belongs to some tree T_ℓ of F. Thus u' is covered by some region of $\mathcal{Q}_{i_j}$ and by some region of $\mathcal{Q}_{i_\ell}$. The fact that $u' \in P_j$ indicates, by definition of P_j, that j must be less than ℓ. Then, by the first step of the construction (Lemma 4.9), u' get connected to T_j in $(F_1 \cap C)/B^C$. Since T_ℓ and T_j are both outgoing, in F u' is connected to some vertex $u \in \partial C$ and T_j is connected to some vertex $v \in \partial C$. By transitivity u is connected to v in $(F_1 \cap C)/B^C$. By the second step of the construction (Lemma 4.10), u is connected to

v in $F' \cap C$. By transitivity again, u' is connected to T_j in $F' \cap C$. □

4.6.3 Length of F'

First we analyze the length increase when gong from F to F_1. Since $u \in P_j$, there exists a region $Q \in \mathcal{Q}_j$ such that $u \in Q$. By definition, Q covers at least one vertex $w \in T_j$. Thus the path added to connect u to T_j in $(F' \cap C)/B^C$ has length at most $2\beta 2^{i_j}$. If $i_j = i'_j$, then the length is at most $4\beta\,\mathrm{length}(T_j \cap C)$. In this case, we charge the length of this path to T_j. Otherwise, it's at most $4\beta\,\mathrm{length}(F \cap C)/\gamma$. In this case, we charge the length of this path to $F \cap C$.

We claim that each tree T_j of F is charged at most $2w - 1$ times by paths added when $i_j = i'_j$. Indeed, whenever T_j is charged, some other tree T_ℓ of F is connected to T_j in $(F' \cap C)/B^C$. After processing descendant clusters of C and adding connections between boundary vertices, T_j is connected to T_ℓ in $F' \cap C$. Since $j < \ell$, the minimal cluster C_j strictly enclosing T_j is either the same as for T_ℓ or one of its descendant clusters. Either way, T_ℓ must contain a vertex of ∂C_j, so there are only at most $2w - 1$ such trees T_ℓ, so T_j is charged at most $2w - 1$ times. Summing over trees T_j, the total charge of those paths is at most $4\beta(2w - 1)\,\mathrm{length}(F)$.

The length charged to $F \cap C$ is at most $4\beta(2w - 1)(3\log_2 m + 1)/\gamma$, since each edge is charged at most $(2w - 1)(3\log_2 m + 1)$ times.

Second, we claim in the second step, going from F_1 to F', each cluster C is charged at most $2w - 1$ times. Indeed, each charge corresponds to two boundary vertices of C being connected by a path, and after $2w - 1$ paths are added, all of ∂C must be connected. Summing over clusters C, the total charge of those paths is at most $\sum_{C \in \mathcal{C}} 2(2w - 1)\rho^C$, which is at most $2(2w - 1)\,\mathrm{length}(G)/\alpha$ by Lemma 4.5.

This completes the proof of Theorem 4.1.

4.7 Proof of Theorem 2.3

Let $\gamma = (3\log_2 m + 1)$, $\beta = \epsilon/8(2w - 1)$, and $\alpha = 2\epsilon^{-1}c(2w - 1)$ where c is the constant specified in Theorem 2.3. Lemma 4.8 implies that the DP takes time $n \log n (\log\log n)^{O(1)}$, which is $O(n \log^2 n)$. By Lemma 4.1, the quality of the output satisfies Theorem 2.3.

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