COMBINATORIAL PROPERTIES OF A ROOTED GRAPH POLYNOMIAL

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Abstract. For a rooted graph G, let $EV(G;p)$ be the expected number of vertices reachable from the root when each edge has an independent probability p of operating successfully. We examine combinatorial properties of this polynomial, proving that G is k-edge connected iff $EV'(G;1) = \cdots = EV^{k-1}(G;1) = 0$. We find bounds on the first and second derivatives of $EV(G;p)$; applications yield characterizations of rooted paths and cycles in terms of the polynomial. We prove reconstruction results for rooted trees and a negative result concerning reconstruction of more complicated rooted graphs. We also prove the norm of the largest root of $EV(G;p)$ in $\mathbb{Q}[i]$ gives a sharp lower bound on the number of vertices of G.

1. Introduction

Graph polynomials have a long history, dating to Birkhoff's use of the chromatic polynomial in an (unsuccessful) attempt to prove the four color theorem [7]. Two other polynomials, the reliability polynomial [13] and the 2-variable Tutte polynomial [8], also encode combinatorial data about the graph (the Tutte polynomial specializes to both the chromatic and reliability polynomials). While the original motivation for the study of these invariants is still important, much of the current interest in the Tutte polynomial is not related to any of its applications. See [9, 15] for some recent combinatorial applications.

It is in this spirit that we continue the study of the expected value polynomial $EV(G;p)$ applied to a rooted graph G, i.e., a graph with a distinguished vertex. The polynomial was introduced in [1] and [2], extended to antimatroids in [19], and applied to rooted graphs in [5, 20]. A closely related polynomial, called *pair connected reliability* by Amin, et. al. in [3, 4, 23, 22] and *network resilience* by Colbourn in [14], is motivated by the reliability polynomial. A similar polynomial has also been defined for (non-rooted) graphs [6, 24].

In this paper, we concentrate on combinatorial properties of the rooted graph and their connection to the polynomial. In the sequel [16], we turn to applications, including practical questions about optimal location for the root for a given graph, randomness and estimation.

Section 2 is concerned with bounds on the first and second derivatives of $EV(G;p)$. Applications of these bounds to the edge connectivity of the graph and graph reconstruction are given. The main result is Theorem 2.7:

Theorem 2.7 *If G is a rooted graph, then G is k-edge-connected if and only if $EV'(G;1) = \cdots = EV^{(k-1)}(G;1) = 0$.*

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Section 3 examines the behavior of $EV(G;p)$ under standard graph-theoretic constructions. We give limits on the possibility of reconstructing G from $EV(G;p)$. When G is a rooted cycle or a rooted path, reconstruction is possible; in almost all other cases, it is not possible.

Theorem 3.4

(1) $EV(G;p) = p + \cdots + p^n$ *if and only if* G *is isomorphic to the rooted path* P_{n-1}.
(2) $EV(G;p) = 2p + \cdots + 2p^{n-1} + (n-1)p^n$ *if and only if* G *is isomorphic to the rooted cycle* C_n.

Conversely, Theorem 3.1 shows that *any* rooted graph is a subgraph of another rooted graph with a linear expected value polynomial. In a slightly different direction, Theorem 3.9 shows that rooted trees can be reconstructed from a *family* of expected rank polynomials.

In Section 4 we give connections between the maximum norm of the zeroes of $EV(G;p)$ in $\mathbb{C}$ and the number of vertices of G. This section is motivated by the study of the roots of the chromatic polynomial, which has connections to statistical physics [11, 21]. Our main result is Theorem 4.3:

Theorem 4.3 *Let* G *be a connected rooted graph with* $n > 1$ *vertices. Suppose that the polynomial* $EV(G;r) = 0$ *for some* $r \in \mathbb{Q}[i]$. *Then* $|r-1| \leq n-1$.

While many of the proofs given here are straightforward (especially those concerning derivatives of the one-variable polynomial), we believe the results are of sufficient interest to warrant further study. These results show that the polynomial encodes meaningful information about the rooted graph, but we also place bounds on how successful such an approach can be (Theorem 3.1).

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2. Derivatives and edge connectivity

Let G be a connected rooted graph with edge set E where each edge has the same independent probability p of being operational. We give two equivalent formulations of the expected rank polynomial, both of which will be important throughout this work. For a non-root vertex v, let $Pr(v)$ denote the probability that v remains connected to the root. The following result appears explicitly in [5] and implicitly in [14] and [3, 4].

Definition 2.1 (Proposition 2.7 of [5])**.** Let G be a rooted graph and let V be all the non-root vertices of G. Then

$$EV(G;p) = \sum_{v \in V} Pr(v).$$

We will also need the following deletion-contraction expansion for the polynomial.

Proposition 2.2 (Deletion-Contraction: Proposition 2.3 of [5])**.** *Let* G *be a rooted graph and let* e *(*$\neq$ *loop) be an edge adjacent to the root. Then*

$$EV(G;p) = (1-p) \cdot EV(G-e;p) + p \cdot EV(G/e;p) + p.$$

We will use the deletion-contraction formula of 2.2 repeatedly in this section. Throughout this section, we will assume an edge e incident to the root is not a loop (loops have no effect on the polynomial).

Example 2.3. We consider the expected rank polynomial $EV(K_n;p)$ for the rooted complete graph. This example is important since any random graph can be thought of as a subgraph of K_n. Obtaining closed form expressions for $EV(K_n;p)$ is difficult; see [4] for one approach. When applying deletion-contraction to K_n, multiple edges arrise, and this gives rise to a recursive formula for $EV(K_n;p)$. (Replacing e by k multiple edges can be thought of as increasing the probability that e succeeds from p to $1-q^k$, where $q=1-p$.) If G' is obtained from the rooted graph G by replacing every edge of G by k parallel edges, then $EV(G') = EV(G;1-q^k)$. The proof of the next proposition follows from this observation.

Proposition 2.4. *Let $q = 1-p$. Then*

$$EV(K_n;p) = \sum_{k=1}^{n-1} \binom{n-1}{k} p^i q^{n-1-k} \left(EV(K_{n-k};1-q^k)+k\right)$$

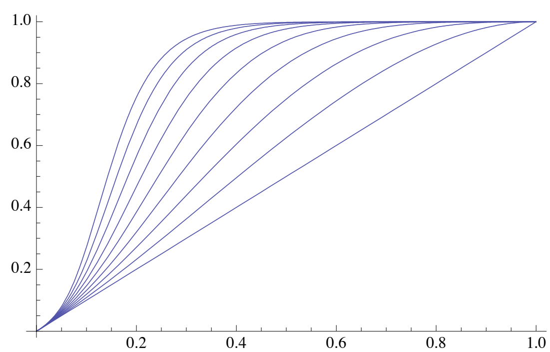

FIGURE 1. Graphs of $EV(K_n;p)/(n-1)$ for $n = 2,\ldots,10$.

For fixed $p>0$, we have $EV(K_n;p)/(n-1) \to 1$ as $n\to\infty$. This follows from a famous result of Erdös and Rényi [18]. If $p >> \log n/n$, then the probability that G is connected approaches 1 as $n\to\infty$. Graphs of $EV(K_n;p)/(n-1)$ for $2\le n\le 10$ appear in Figure 1.

We now turn our attention to the derivative $EV'(G;p)$, which is closely related to the connectivity of the rooted graph G. We begin by deriving sharp bounds on the size of $EV'(G;p)$ and $EV''(G;p)$.

We omit the straightforward proof of the lemma.

Lemma 2.5. *Let G be a rooted graph with an edge e incident to the root. Then for all $p\in[0,1]$, $1+EV(G/e;p) \ge EV(G-e;p)$, with equality possible only when $p=1$.*

It is easy to see that $EV'(G;p)\ge 0$ for all $0\le p\le 1$: increasing p increases the expected number of vertices reachable from the root, so $EV(G;p)$ is an increasing function. We now give sharp upper and lower bounds on $EV'(G;p)$ and an upper bound on the second derivative $|EV''(G;p)|$.

Proposition 2.6. *Let G be a rooted graph with n edges. Then for all $p \in [0,1]$,*

(1) $EV'(G;p) \geq 0$. *This inequality is strict if G has an edge incident to the root and $p < 1$.*
(2) $EV'(G;p) \leq n(n+1)/2$.
(3) $|EV''(G;p)| \leq (n-1)n(n+1)/3$.

Proof. We prove (2) – the proofs of (1) and (3) are similar.

Proof of (2): We proceed by induction on n, and the base case $n = 0$ is trivial. When $n > 0$, we differentiate the formula 2.2:

$$\begin{aligned} EV'(G;p) &= (1 + EV(G/e;p)) - EV(G-e;p) \\ &\quad + pEV'(G/e;p) + (1-p)EV'(G-e;p) \end{aligned}$$

If G has no edge e incident to the root, $EV'(G;p) = 0$. Otherwise, we examine each term in this formula:

First, note that $1 + EV(G/e;p) \leq n$ since $EV(G/e;1) = n-1$ and $EV(G;p)$ is an increasing function. Also, $EV(G-e;p) \geq 0$ is clear. Finally,

$$pEV'(G/e;p) + (1-p)EV'(G-e;p) \leq (n-1)n/2$$

by induction. Putting the pieces together gives $EV'(G;p) \leq n(n+1)/2$.

□

Notes

(1) The lower bound 2.6(1) is sharp for all 2-edge-connected graphs at $p = 1$. For the rooted cycle C_n,

$$EV'(C_n;p) = \sum_{k=1}^{n-1} 2kp^{k-1} - n(n-1)p^{n-1},$$

so $EV'(C_n;1) = 0$.

(2) The upper bound 2.6(2) is also sharp for rooted paths with n edges. If P_{n+1} denotes the rooted path on n edges with root located at a leaf, then

$$EV(P_{n+1};p) = \sum_{k=1}^{n} p^k,$$

so $EV'(P_{n+1};1) = \sum_{k=1}^{n} k = n(n+1)/2$. The converse is also true when $p = 1$: If $EV'(G;1) = n(n+1)/2$, then G is a rooted path (Lemma 3.3(1)).

(3) Note that it is not possible to bound $EV'(G;p)$ in terms of the number of vertices: if G is a graph with two vertices joined by k edges then $EV(G;p) = 1-(1-p)^k$, so $EV'(G;0) = k$.

(4) The bounds of 2.6(3) are sharp for paths (upper bound) and cycles (lower bound) with n edges (P_{n+1} and C_n), again at $p = 1$. We will also prove a converse for the lower bound: If $EV''(G;1) = -(n-1)n(n+1)/3$, then G is the rooted cycle C_n (Lemma 3.3(2)).

It is possible to derive bounds for even higher derivatives in this fashion, but these bounds will not be sharp, in general, with the exception of the lower bound for the third derivative. Further, the expected value polynomials of paths and cycles do not have maximal and minimal derivatives of all orders at $p = 1$.

Note that the second derivative of the deletion-contraction formula 2.2 simplifies when $p = 1$:

$$EV''(G;1) = 2EV'(G/e;1) - 2EV'(G-e;1) + EV''(G/e;1).$$

This allows us to rewrite the upper bound in 2.6(3) in terms of the number of vertices of G, since G/e has one fewer vertex than G.

Recall that G is k-edge connected if removing fewer than k edges from G cannot disconnect G. The next result shows that k-edge connectivity is determined by $EV(G;p)$.

Theorem 2.7. *If G is a rooted graph, then G is k-edge-connected if and only if $EV'(G;1) = \cdots = EV^{(k-1)}(G;1) = 0$.*

Proof. From 2.1, we have

$$EV(G;p) = \sum_{v \in V} Pr(v),$$

where $Pr(v)$ is the probability that v is connected to the root. Fix v and let $S_1, \ldots, S_n$ be the minimal subsets of E that, when removed, disconnect v from the root. Let $F(S)$ be the probability that all edges in S fail. Then we can compute $Pr(v)$ in terms of $F(S)$ via inclusion-exclusion:

$$Pr(v) = 1 - F(S_1) - \cdots - F(S_n) + F(S_1 \cup S_2) + \cdots \pm F(S_1 \cup \cdots \cup S_n).$$

Now $F(S_i) = (1-p)^{|S_i|}$, so $Pr(v) = 1 + \sum_{j=1}^{m} a_j(1-p)^j$. Clearly, if k is the size of the smallest S_i, then $a_1 = \cdots = a_{k-1} = 0$ and $a_k \neq 0$ (in fact, we must have $a_k < 0$). Finally, summing over all vertices gives the result.

□

As a quick check, note that $EV'(T;1) > 0$ for any tree having $n > 0$ edges, so Theorem 2.7 shows any tree is 1-edge connected. For the cycle C_n, we have $EV'(C_n;1) = 0$, (see the remarks immediately following 2.6), but $EV''(C_n;1) = -(n-1)n(n+1)/3$, so the theorem gives a verification that cycles are 2-edge connected.

When G is not rooted, Proposition 2.2 of [6] shows that $EV'(G;1) = 0$ iff G is connected, where $EV(G;p)$ is defined using the matroid (cycle) rank function. In this case, the value of $|EV'(G;1)|$ is just the number of isthmuses of G. Thus, Theorem 2.7 is a rooted generalization of this result.

3. Reconstructing graphs and an embedding theorem

$EV(G;p)$ is defined via connectivity; thus, it is not surprising that graph reconstruction is not possible using this invariant. The next theorem, one of the main results of this section, is a negative reconstruction result.

Theorem 3.1. *Let G be a rooted graph. Then there is a rooted graph G' such that G is an induced subgraph of G' and $EV(G') = kp$ for some positive integer k.*

Proof. Let $EV(G;p) = a_1p + a_2p^2 + \cdots + a_np^n$, where $a_i \in \mathbb{Z}$, and $a_n \neq 0$. We first show how to find a graph H_1 with the property that $EV(G \oplus H_1;p)$ has degree less than n. We then iterate this procedure, eventually producing a graph $G' = G \oplus H_1 \oplus \cdots H_{n-1}$ so that $EV(G';p)$ is a linear polynomial.

Case 1. $a_n < 0$. Let H_1 be the direct sum of a_n copies of the path P_{n+1}, the path with n edges, with each path rooted at a vertex of degree 1. Since $EV(P_{n+1}; p) = \sum_{k=1}^{n} p^k$, we have find that the degree of $EV(G \oplus H_1)$ is at most $n-1$.

Case 2. $a_n > 0$. First attach k n-cycles to the root of G, where $k(n-1) > a_n$, and call the new graph G_1. Now $EV(G_1; p) = b_n p^n + \cdots$ has degree n, and $b_n = a_n - k(n-1) < 0$, by construction. Now proceed as in case 1.

Now iterate this procedure to produce rooted graphs $H_2, H_3, \ldots$ so that the degree of $EV(G \oplus H_1 \oplus \cdots \oplus H_k)$ is at most $n-k$. This process will terminate when $k = n-1$.

□

Example 3.2. We apply (a variation of) the procedure described in the proof of Theorem 3.1 to the rooted cycle C_3. First, note that $EV(C_3; p) = 2p + 2p^2 - 2p^3$. We attach a tree H_1 with $EV(H_1; p) = p + p^2 + 2p^3$, as in Figure 2. This gives $EV(C_3 \oplus H_1; p) = 3p + 3p^2$.

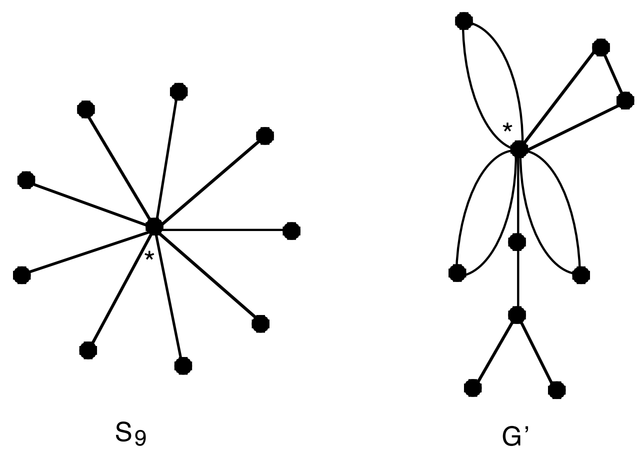

FIGURE 2. $EV(S_9; p) = EV(G'; p) = 9p$.

Now let $H_2 = C_2 \oplus C_2 \oplus C_2$, so $EV(H_2; p) = 6p - 3p^2$. Thus $EV(G'; p) = 9p$, where $G' = C_3 \oplus H_1 \oplus H_2$.

For example, if we let A_k denote the rooted graph formed by k parallel edges, then $EV(C_2 \oplus C_2 \oplus C_2 \oplus C_2; p) = EV(A_3 \oplus A_3 \oplus C_3; p) = 8p - 4p^2$.

Given Theorem 3.1, it is quite easy to construct non-isomorphic graphs with the same expected value polynomial. However, for certain classes of graphs, G can be uniquely reconstructed from $EV(G; p)$. We now show how rooted cycles are completely determined (within the class of all rooted graphs) by these polynomials. Recall that P_{n+1} denotes the rooted path with n edges.

Lemma 3.3. *Let G be a rooted graph with n edges.*

(1) $EV'(G; 1) = (n+1)n/2$ *if and only if G is isomorphic to P_{n+1}.*
(2) $EV''(G; 1) = -(n+1)n(n-1)/3$ *if and only if G is isomorphic to C_n.*

Proof. We prove (1) and leave the similar proof of (2) to the reader. From the remarks following the proof of Proposition 2.6(2), $EV'(P_{n+1};1) = n(n+1)/2$.

For the converse, we use induction. If G has 1 edge, then there is nothing to prove. Now suppose $n > 1$ and $EV'(G;1) = (n+1)n/2$. If G has no edges incident to the root, then $EV'(G;1) = 0$, so we may assume e is incident to the root. Then, as in the proof of 2.6(2), we have $EV'(G;1) = 1 + EV(G/e;1) - EV(G-e;1) + EV'(G/e;1)$.

Now $EV(G/e;1) \le n-1$ (since $G-e$ has $n-1$ edges) and $EV(G-e;1) \ge 0$. Thus,

$$EV'(G/e;1) = EV'(G;1) - EV(G/e;1) + EV(G-e;1) - 1,$$

which gives $EV'(G/e;1) \ge (n+1)n/2 - (n-1) - 1 = n(n-1)/2$. By Proposition 2.6(2), we have $EV'(G/e;1) = n(n-1)/2$, which forces each of the inequalities given above to be equalities. Thus, $EV(G-e;1) = 0$, so e is the only edge incident to the root of G, and $EV(G/e;1) = n-1$, so G/e is connected. Furthermore, since $EV'(G/e;1) = n(n-1)/2$, we have G/e is isomorphic to the path P_n, by induction.

Now we have a rooted graph G with exactly one edge e incident to the root such that G/e is the path P_n. This forces G to be the path P_{n+1}.

□

An immediate consequence of Lemma 3.3 is the following.

Theorem 3.4.
1. *$EV(G;p) = p + \cdots + p^n$ if and only if G is isomorphic to the rooted path P_{n-1}.*
2. *$EV(G;p) = 2p + \cdots + 2p^{n-1} + (n-1)p^n$ if and only if G is isomorphic to the rooted cycle C_n.*

Obviously, for any positive integer k, it is possible to produce k non-isomorphic rooted trees, all sharing the same expected value polynomial. On the other hand, it is possible to uniquely reconstruct a rooted tree from a *sequence* of expected rank polynomials. We begin with a definition.

Definition 3.5. Let G be a rooted graph. Then the *expected rank k* polynomial is defined by

$$EV_k(G;p) = \sum_{A \subseteq E : r(A) = k} p^{|A|}(1-p)^{|E|-|A|},$$

where $r(A)$ is the number of vertices in the component of A containing the root (not including the root).

Thus, $EV_k(G;p)$ is the probability that exactly k vertices are connected to the root. The proof of the next proposition is immediate.

Proposition 3.6. *Let G be a rooted graph with $n+1$ vertices. Then*
1. $\sum_{k=1}^{n} EV_k(G;p) = 1$,
2. $EV(G;p) = \sum_{k=1}^{n} k \cdot EV_k(G;p)$.

To keep track of this sequence of expected rank k polynomials, it is convenient to introduce a 2-variable generating function.

Definition 3.7. Let T be a rooted tree, and let $X(T)$ be a tree with a single edge e adjacent to the root such that $X(T)/e = T$. Then define $F(T;p,q)$ recursively as follows:

$$\begin{aligned} F(\bullet) &= 1 \\ F(X(T)) &= q + pF(T) \\ F(T_1 \oplus T_2) &= F(T_1)F(T_2) \end{aligned}$$

The connection between the generating function $F(T; p, q)$ and the sequence of rank k expected rank polynomials $EV_k(T; p)$ is made explicit in the next proposition.

Proposition 3.8. *$F(T; p, q)$ is uniquely recoverable from the sequence of polynomials $EV_0, \ldots, EV_n$, where EV_k is the probability that exactly k vertices are connected to the root.*

Proof. Let $EV_k(p) = p^k g_k(p)$. Then set $q = 1 - p$, so $p = 1 - q$, and it is easy to show $F(T) = \sum_k p^k g_k(1 - q)$. Thus, we can recover $F(T)$ from the sequence of polynomials, and this operation is easily invertible.

$\square$

We now prove that rooted trees can be uniquely reconstructed from their sequence of rank k expected rank polynomials $\{EV_0, \ldots, EV_n\}$.

Theorem 3.9. *Let T be a rooted tree. Then $F(T(p, q))$ uniquely determines T up to isomorphism.*

Proof. It suffices to show that for all T, $F(X(T))$ is irreducible over $\mathbf{Z}[p, q]$. The result then follows by induction: if $F(T; p, q)$ factors, we reconstruct the rooted trees corresponding to the factors inductively. If $F(T; p, q)$ is irreducible, we will have $T = X(T')$ for some rooted tree T', and $F(T'; p, q) = p^{-1}(F(T; p, q) - q)$, so we can reconstruct T' (and hence, T) inductively again.

Now write $F(T) = q + pG(p, q)$ for some polynomial $G(p, q)$ and suppose that $F(X(T)) = AB$. Then $F(X(T)) = (1 + pA')(q + pB')$ for some two-variable polynomials A' and B'. If $q \mid B'$, then $q \mid (q + pB')$, so $q \mid F(X(T))$, which cannot be the case since $F(X(T))$ has exactly one pure p term: p^n, corresponding to all n edges operating successfully.

Hence q does not divide B' and we let cp^α with $c \neq 0$ be the pure p term in B' of lowest degree. Then $F(X(T))$ contains a term cp^α that cannot be canceled by $p^2A'B'$. As a result, $A' = 0$ and the factorization is trivial.

$\square$

The sequence $\{EV_0, \ldots, EV_n\}$ is equivalent to the (greedoid) Tutte polynomial of a rooted tree, which encodes information about the number of rooted subtrees of size k with exactly l leaves. More information about rooted tree reconstruction from this version of the Tutte polynomial can be found in [2, 12]. (Unrooted tree reconstruction is not possible in general – see [17].)

4. Zeroes of the polynomial

Proposition 3.1 of [20] shows that the largest rational root of $EV(G; p)$ is a lower bound on the number of vertices (including the root). We generalize this result now, extending the bound to the absolute value of the largest rational root.

Proposition 4.1. *Let G be a connected rooted graph with $n > 1$ vertices. Suppose the polynomial $EV(G; r) = 0$ for some $r \in \mathbb{Q}$. Then $|r - 1| \leq n - 1$.*

Proof. Let $f(p) = EV(G; p+1)$. Then we can write

$$f(p) = (a - bp)g(p)$$

where $a/b = r - 1$ and $g \in \mathbb{Z}[p]$. $f(0) = a \cdot g(0) = n - 1$, so $a \mid (n-1)$; and since $|a/b| \leq |a|$, $|r - 1| \leq n - 1$.

□

We can extend this result further, to the case where the root $r \in \mathbb{Q}[i]$.

Proposition 4.2. *Let G be a connected rooted graph with $n > 1$ vertices. Suppose the polynomial $EV(G; r) = 0$ for some $r \in \mathbb{Q}[i] - \mathbb{Q}$. Then $|r - 1|^2 \leq n - 1$.*

Proof. Again, let $f(p) = EV(G; p+1)$ and write $(a + bi)/c = r - 1$ for integers a, b and c. Then

$$f(p) = (a^2 + b^2 - 2a(cp) + (cp)^2)g(p)$$

where $g(p) \in \mathbb{Z}[p]$. Since $n - 1 = f(0) = (a^2 + b^2)g(0)$, and $a^2 + b^2 \mid n - 1$. We have $a^2 + b^2 = |c(r-1)|^2 \geq |r - 1|^2$, so $|r - 1|^2 \leq n - 1$.

□

Putting Propositions 4.1 and 4.2 together gives the following.

Theorem 4.3. *Let G be a connected rooted graph with $n > 1$ vertices. Suppose that the polynomial $EV(G; r) = 0$ for some $r \in \mathbb{Q}[i]$. Then $|r - 1| \leq n - 1$.*

All of these bounds are sharp. For the rational roots of Proposition 4.1, let G_1 be a graph with one vertex connected to the root by two edges and k vertices connected by one edge. Then $EV(G_1; p) = (k + 2)p - p^2$, which has a root at $p = k + 2$. For the lower bound, if we let G_2 be a tree with polynomial $kp + p^2$, then $EV(G_2; p)$ has a root at $p = -k$.

For the imaginary rational roots of 4.2, let $a^2 + b^2 = c^2$ be a Pythagorean triple and let T be a tree with polynomial $((a-1)^2 + b^2)p + 2(a-1)p^2 + p^3$. Then $EV(T; p)$ has roots at $p = 1 - a \pm bi$, and T has $((a-1)+1)^2 + b^2 + 1 = c^2 + 1$ vertices and $|1 - a \pm bi - 1| = c$.

The next result is applicable to any polynomial $f(p)$ with positive integer coefficients and $f(0) = 0$.

Proposition 4.4. *Let T be a tree with $n > 1$ vertices. Suppose the polynomial $EV(T; z) = 0$ for some $z \in \mathbb{C}$. Then $|z| \leq n - 2$.*

Proof. $EV(T; p) = n_1 p + n_2 p^2 + \cdots + n_k p^k$ for positive integers n_j. Let $C = n_1 + \cdots + n_{k-1}$. Then $C \leq n - 2$, since $n_1 + \cdots + n_k = n - 1$.

When $C = 0$, $EV(T; p) = n_k p^k$, which has zeros only at $p = 0$. Otherwise we can assume that $C \geq 1$. Suppose to the contrary that $|z| > C$. Then

$$|-n_k z^k| > C|z^{k-1}| > n_1|z| + \cdots + n_{k-1}|z^{k-1}| > |n_1 z + \cdots + n_{k-1} z^{k-1}|$$

and z is clearly not a zero.

□

When T is a tree, Proposition 4.4 allows us to drop the restriction that $r \in \mathbb{Q}[i]$.

Corollary 4.5. *Let T be a tree with $n > 1$ vertices. Suppose the polynomial $EV(T; p)$ has a zero at $p = z \in \mathbb{C}$. Then $|z - 1| \leq n - 1$.*

Unfortunately, this bound does not extend to all graphs and all complex zeros. For example, let $G = K_4 \oplus T$, where T is a tree such that $EV(T;p) = p + p^2 + p^3 + p^4 + p^5 + 5p^6$. Then $EV(G;p) = 4p + 7p^2 + p^3 - 20p^4 + 22p^5 - p^6$, which has a zero near $p = 21$. G, however, has only 14 vertices and 16 edges.

Similar constructions work with larger complete graphs, where we attach the smallest tree that will make the leading coefficient of $EV(G;p)$ equal to -1.

It would be interesting to determine what other restrictions exist on zeros of the polynomial. This is similar to much of the current research on the chromatic polynomial of a graph [11], [21].

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